A BIJECTIVE PROOF FOR RECIPROCITY THEOREM

SHINNYIH HUANG, ALEXANDER POSTNIKOV

Abstract. In this paper, we study the graph polynomial that records spanning rooted forests f_G of a given graph. This polynomial has a remarkable reciprocity property. We give a new bijective proof for this theorem which has Prüfer coding as a special case.

1. Introduction

A spanning tree T in some graph G is a connected acyclic subgraph of G that includes all vertices in $V(G)$. Calculating the number $t(G)$ of spanning trees for some graph G is one of the typical questions we will ask. For example, when G is a complete graph K_n, $t(K_n) = n^{n-2}$. There are several methods to calculate $t(G)$, such as the matrix-tree theorem and Prüfer coding.

In this paper, we study some graph polynomial f_G that records the spanning trees of the extended graph $\widetilde{G}$ of graph G. This polynomial can be used to compute the spanning tree of some complex graphs easily. For example, let $\Gamma = \Gamma\left(G; G_1, \ldots, G_k\right)$ be the graph that is obtained by substitution of graphs $G_1, \ldots, G_k$ instead of a vertices of a graph G. Then we can easily obtain f_Γ by f_G and f_{G_i}, for $1 \le i \le k$.

In fact, the polynomial f_G possess the remarkable property of reciprocity. A. Renyi [9] gives an inductive proof for this reciprocity theorem. I. Pak and A. Postnikov [1] also give an inductive proof. Throughout this paper, we present a new bijective proof for the reciprocity theorem. One interesting fact is that the map we used in the bijection is Prüfer coding when G is a complete graph.

This paper is organized as follows: In section 2, we define the graph polynomial f_G to enumerate spanning trees in $\widetilde{G}$. In section 3, we show the reciprocity theorem for f_G and defined some tools for the future bijective proof. In section 4, we define two maps ϕ and ψ to show the bijection between **A** and **B**. Finally, in section 5, we use this bijective coorespondence to prove the reciprocity theorem of f_G.

2. Graph Polynomials for Spanning Trees

Suppose that $G = (V, E)$ is a graph with vertices $1, \ldots, n$, where $|V| = n$. Let $0 \notin V$ and $\widetilde{V} := V \cup \{0\}$. We say the *extended graph* $\widetilde{G}$ of G is a graph on the set $\widetilde{V}$ obtained by adding edges $\{0, v\}$ to G for all vertices $v \in V$. Clearly, if G is a complete graph K_n with n vertices, then $\widetilde{G}$ is a complete graph K_{n+1} with $n+1$ vertices. We denote the set of all *spanning trees* in G as $\mathcal{T}_G$, i.e. all acyclic connected subgraphs in G which contain all the vertices of G.

First of all, we assign variables x_i to i, for all $1 \le i \le n$. For any spanning tree T in $\mathcal{T}_G$, define a function $m(T)$ associated to T:

$$m(T) = \prod_{v \in V} x_v^{\rho_T(v)-1}, \tag{2.1}$$

where $\rho_T(v)$ denotes *degree* of the vertex v in the tree T, i.e. the number of edges adjacent to the vertex v.

Now, we set the graph polynomial t_G to be,

$$t_G := \sum_{T \in \mathcal{T}(G)} m(T).$$

Let us associate the variable x to vertex 0. Then, the graph polynomial f_G of variables x and x_v, for all $v \in V$ is defined as follows:

$$f_G := t_{\widetilde{G}} = \sum_{T \in \mathcal{T}_{\widetilde{G}}} m(T). \tag{2.2}$$

We denote $V = \{1, \ldots, n\}$ and $f_G = f_G(x; x_1, \ldots, x_n)$.

It is easy to see that the spanning trees in $\mathcal{T}_{\widetilde{G}}$ correspond to *spanning rooted forests* in G, i.e. acyclic subgraphs in G containing all vertices in V, with a root chosen in each component. In particular, the two polynomials t_G and f_G possess the following identity:

$$t_G(x_1, \ldots, x_n) \cdot (x_1 + \cdots + x_n) = f_G(0; x_1, \ldots, x_n). \tag{2.3}$$

An short proof for Eq.(2.3) is provided in Igor Pak and A. Postnikov [1].

The graph polynomial f_G has two important properties that allow us to compute the number of spanning rooted forests for certain graph. The first property is the composition of graphs. Let G_1 and G_2 be two graphs on disjoint sets of vertices, and $G_1 + G_2$ be the disjoint union of the graphs. We associate variable x to the root 0, variables $y_1, \ldots, y_{r_1}$ to the vertices of G_1, and variables $z_1, \ldots, z_{r_2}$ to the vertices of G_2. Then the following formula holds:

$$f_{G_1+G_2}(x; y_1 \ldots, y_{r_1}, z_1 \ldots, z_{r_2}) = x \cdot f_{G_1}(x; y_1 \ldots, y_{r_1}) \cdot f_{G_2}(x; z_1 \ldots, z_{r_2}).$$

One can prove the above equation by some simple arguments.

3. Reciprocity Theorem For Polynomials f_G

A graph $\overline{G} = (V, \overline{E})$ is called the *compliment* of some graph $G = (V, E)$ if $\overline{E} = \binom{V}{2} \backslash E$. That is to say, $e \in \overline{E}$ iff $e \notin E$. The graph polynomials f_G possess the following *reciprocity property*:

$$f_G(x; x_1, \ldots, x_n) = (-1)^{n-1} \cdot f_{\overline{G}}(-x - x_1 - \cdots - x_n; x_1, \ldots, x_n). \tag{3.1}$$

The case that $x_1 = \cdots = x_n = 1$ for (3.1) was found by S. D. Bedrosian [2] and A. Kelmans.

Before we give the bijective proof for Eq.(3.1), we first introduce some notation.

First of all, let $\acute{F}_G$ be a spanning tree of some extended graph $\widetilde{G}$ with root 0 and vertices $1, \ldots, n$ so that F_G is a spanning rooted forest of G. It is easy to show that for any vertex u of G, there is a unique path from u to root 0. Therefore, we can assign a direction to every edge in $\acute{F}_G$ such that each arrow points toward the root 0. This implies that every vertex $u \neq 0$ has outdegree 1. For convention, in this paper, when we say graphs $\acute{F}_G \in \mathcal{T}_{\widetilde{G}}$ or F_G, we always consider it as a directed graph, and thus for every $u \neq 0$, there is a unique directed edge $(u, v) \in E(\acute{F}_G)$. In addition, a vertex u is the *child* of vertex u_1 if there is a directed path from u to u_1 in $\mathcal{T}_{\widetilde{G}}$.

Secondly, we say that a *valid pair* of some tree $\acute{F}_{K_n}$ is a pair $(u,v) \in F_{K_n}$, and $\mathbf{Z}_{G,\acute{F}_{K_n}}$ is a subset of valid pairs of $\acute{F}_{K_n}$ such that

$$\mathbf{Z}_{G,\acute{F}_{K_n}} = \{(u,v) : (u,v) \notin E(\overline{G}), (u,v) \in E(F_{K_n})\}. \tag{3.2}$$

Now, given a subset $\mathbf{C}$ of all valid pairs not in $\mathbf{Z}_{G,\acute{F}_{K_n}}$, we define an *operational set* $\mathcal{O}_{G,\acute{F}_{K_n},\mathbf{C}}$ as follows:

$$\mathcal{O}_{G,\acute{F}_{K_n},C} = \mathbf{C} \cup \mathbf{Z}_{G,\acute{F}_{K_n}}. \tag{3.3}$$

One can see that for a spanning tree $\acute{F}_{K_n}$ and graph $G \in K_n$, there could be many possible operational sets. An example is in figure 1.

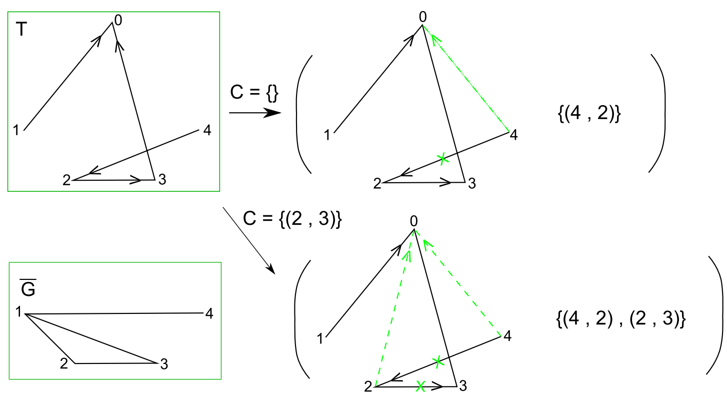

FIGURE 1. For $\acute{F}_{K_n}$ and $\overline{G}$ as above, we have two possible operational sets for $\acute{F}_{K_n}$. (The green marks are the graph after we apply all the pair in the operation sets to $\acute{F}_{K_n}$.)

Now, for any $\acute{F}_{\overline{G}}$, suppose its induced subgraph $F_{\overline{G}}$ in K_n has k connected components. We say a *weight sequence* $\mathcal{W}_{\acute{F}_{\overline{G}}}$ of $\acute{F}_{\overline{G}}$ is

$$\mathcal{W}_{\acute{F}_{\overline{G}}} = (w_1, \ldots, w_{k-1}), \tag{3.4}$$

where $w_j \in \{0, 1, \ldots, n\}$, for $1 \le j \le k-1$. By convention, if $k = 1$, we set $W_{T_{\tilde{\overline{G}}}}$ to be empty. Therefore, there are $(n+1)^{k-1}$ possible weight sequences for spanning tree $\acute{F}_{\overline{G}}$ that has k connected compoenents in $F_{\overline{G}}$.

Given a graph $G \in K_n$, let $\mathbf{A}$ be the set of all possible pairs $\left(\acute{F}_{K_n}, \mathcal{O}_{G,\acute{F}_{K_n},C}\right)$ and $\mathbf{B}$ be the set of all possible pairs $\left(\acute{F}_{\overline{G}}, \mathcal{W}_{\acute{F}_{\overline{G}}}\right)$. In the following section, we show a bijection between $\mathbf{A}$ and $\mathbf{B}$.

4. Bijection Between A to B

Suppose that G is a graph with n vertices labeled $1, \ldots, n$ where each vertex i is associated to a variable x_i, for $1 \le i \le n$. For the root in the extended graph, we assign variable x to root 0. We first construct a map ϕ from $\mathbf{A}$ to $\mathbf{B}$.

Definition 4.1. Given a pair $\left(\acute{F}_{K_n}, \mathcal{O}_{G,\acute{F}_{K_n},C}\right) \in A$, the map ϕ outputs a pair $(\acute{F}, \mathcal{W})$ and is defined as follows:

Let S be the set of vertices u in $\acute{F}_{K_n}$, where the directed edge $(u, v) \in E(\acute{F}_{K_n})$ is a pair in $\mathcal{O}_{G,\acute{F}_{K_n},C}$ or $v = 0$. Construct an empty sequence $\mathcal{W}$ and a graph $\acute{F}$ which is a duplicate of $\acute{F}_{K_n}$.

WHILE $|S| > 1$,

1: Suppose there is a leaf $u' \neq 0$ in $\acute{F}_{K_n}$ such that the edge $(u', v') \in E(\acute{F}_{K_n})$ is not in S. We remove u' and (u', v') from $\acute{F}_{K_n}$.
2: Repeat step 1 until every leaf $u \neq 0$ in $\acute{F}_{K_n}$ is also in S. Let M to be the set of all these vertices.
3: Delete the largest vertex u^* in M and the directed edge (u^*, v^*) in $\acute{F}_{K_n}$. We set S to be $S \backslash \{u^*\}$, and add v^* to the end of the sequence $\mathcal{W}$.
4: Remove edge (u^*, v^*) and add edge $(u^*, 0)$ to $\acute{F}$.

RETURN $(\acute{F}, \mathcal{W})$.

An example of this algorithm is in figure 2. In the following proposition, we prove that ϕ is well-defined.

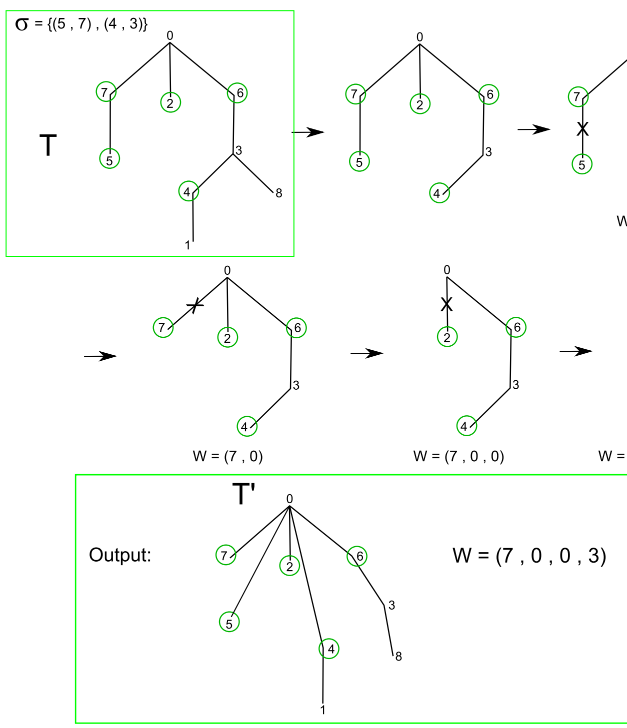

FIGURE 2. Input: $T = \acute{F}_{K_n}$ and $\mathcal{O} = \mathcal{O}_{G,\acute{F}_{K_n},C} = \{(5, 7), (2, 3)\}$,
Output: $T' = \acute{F}_{\overline{G}}$ and $\mathcal{W} = \mathcal{W}_{\acute{F}_{\overline{G}}} = \{7, 0, 0, 3\}$

Proposition 4.2. *The map ϕ is a well-defined map from* **A** *to* **B**.

Proof. It is easy to see that all the steps in WHILE loop work. Now, we show that $\acute{F}$ is a spanning tree of $\widetilde{K_n}$ after each step 4. We proceed this by induction.

Initially, $\acute{F} = \acute{F}_{K_n}$ is a tree. Suppose that at some step 4, we delete edge (u^*, v^*) and add edge $(u^*, 0)$ to the spanning tree $\acute{F} \in \mathcal{T}_{\widetilde{K_n}}$. Furthermore, since for any vertex $u \neq 0$, u and root 0 is connected in graph $\acute{F}$, it remains connected after we change some edge (u^*, v^*) to edge $(u^*, 0)$. Since $|E(\acute{F})| = n$, $\acute{F}$ is always a spanning tree of $\widetilde{K_n}$ after any step 4.

Now, from (3.3), we know that $\mathbf{Z}_{G,\acute{F}_{K_n}} \in \mathcal{O}_{G,\acute{F}_{K_n},C}$ and all the edges (u, v) in the operational set $\mathcal{O}_{G,\acute{F}_{K_n},C}$ became $(u, 0)$ in the output graph $\acute{F}$. Thus, every edge in $E(F)$ is also in $E(\overline{G})$, and $\acute{F}$ is a spanning tree of $\widetilde{\overline{G}}$.

Finally, we show that W is a weight sequence of $\acute{F}$. Clearly, S is the set of all roots in the spanning rooted forest F. Since the WHILE loop ends when $|S| = 1$, there are totally $|S| - 1$ elements added to the sequence W. Consequently, W satisfies the length requirement in Eq.(3.4).

The above arguments tell us that $(\acute{F}, \mathcal{W}) \in B$ as desired. □

We now give a map ψ from $\mathbf{B}$ to $\mathbf{A}$.

Definition 4.3. Given a pair $\left(\acute{F}_{\overline{G}}, \mathcal{W}_{\acute{F}_{\overline{G}}}\right) \in \mathbf{B}$, the map ψ outputs $\left(\acute{F}^*, \mathcal{O}\right)$ and is defined as follows:

Assume that the forest $F_{\overline{G}}$ has k connected components and the associated weight sequence $\mathcal{W}_{\acute{F}_{\overline{G}}} = (w_1, \ldots, w_{k-1})$. Create a tree $\acute{F}^* = \acute{F}_{\overline{G}}$, sequence $\mathcal{W}_{\acute{F}^*} = \mathcal{W}_{\acute{F}_{\overline{G}}}$, and an empty set $\mathcal{O}$. Let R be the set of roots in $F_{\overline{G}}$.

WHILE the length of $\mathcal{W}_{\acute{F}^*}$ is larger than 0.

1: We choose the first element w in the sequence $\mathcal{W}_{\acute{F}^*}$. Let u be the largest vertex in R such that w_i is not u nor a child of u in $\acute{F}^*$, for any w_i in $\mathcal{W}_{\acute{F}^*}$. Delete the element w from the sequence $\mathcal{W}_{\acute{F}^*}$ and u from the set R.

2: Remove the edge $(u, 0)$ and add the edge (u, w) to the graph $\acute{F}^*$. If $w \neq 0$, we add pair (u, w) to the set $\mathcal{O}$, i.e. $\mathcal{O} = \mathcal{O} \cup \{(u, w)\}$.

RETURN $(\acute{F}^*, \mathcal{O})$.

An example of this mapping ψ is in figure 3. In the following lemma, we prove that ψ is well-defined.

Proposition 4.4. *The map ψ is a well-defined map from $\mathbf{B}$ to $\mathbf{A}$.*

Proof. We first show that at any stage, the set R and graph $\acute{F}^*$ satisfy the following properties:

(1) $\acute{F}^*$ is a spanning tree of $\widetilde{K_n}$, i.e. F^* is a sapnning rooted forest of K_n.
(2) R is the sets of roots of forest F^*.

We proceed by induction on the number of loops. Initially, R is the set of all the roots in forest $F_{\overline{G}}$, and $\mathcal{W}_{\acute{F}^*}$ is a sequence of length $k - 1 = |R| - 1$. Moreover, at each step 1, we remove an element in $\mathcal{W}_{\acute{F}^*}$ and an element in R. Thus, the length of sequence $\mathcal{W}_{\acute{F}^*}$ is always $|R| - 1$.

Now, suppose at some stage, we have that properties (1) and (2) hold and sequence $\mathcal{W}_{\acute{F}^*} = \{w'_1, \ldots, w'_{k_1-1}\}$, where $k_1 = |R|$. During step 1, since there are k_1 connected components in F^*, there exists at least one connected component that contains no elements in $\mathcal{W}_{\acute{F}^*}$. Consider the compoenent with the largest root u

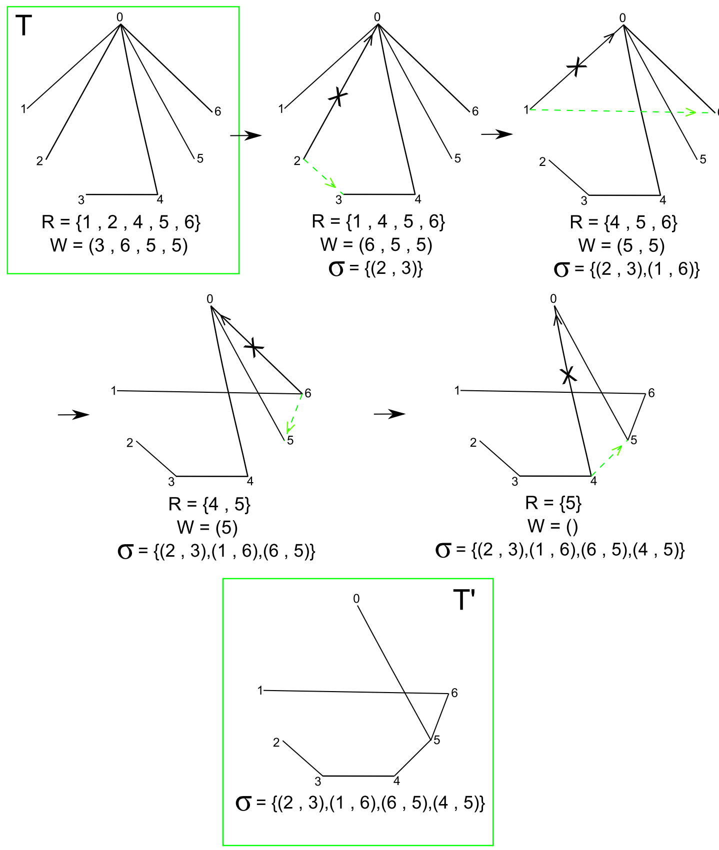

FIGURE 3. Input: $T = \acute{F}_{\overline{G}}$ and $W = \mathcal{W}_{\acute{F}_{\overline{G}}} = \{3, 6, 5, 5\}$, Output: $T' = \acute{F}^*$ and operational set $\sigma = \mathcal{O} = \{(2,3), (1,6), (6,5), (4,5)\}$. (R is the set of current roots.)

that meets this condition. It is not hard to see that for any $1 \leq i \leq k_1 - 1$, w'_i is not u nor a child of u. Consequently, step 1 works.

For step 2, by the choice of vertex u, we have w'_i and u are not connected in F^*. Suppose $\acute{F}^*$ becomes cyclic after we delete edge $(u, 0)$ and add edge (u, w'_1) to this graph. This implies that there is a cycle containing edge (u, w'_1). It is not possible since vertices u and w'_1 would be connected in F^* before we add edge (u, w'_1).

The above arguments show that after step 1 and 2, $\acute{F}^*$ remains acyclic, and is a spanning tree of $\widetilde{K_n}$. Futhermore, after step 2, since u is no longer a root, R remains as the set of all roots in F^*. As a result, properties (1) and (2) always hold.

Finally, we need to show that $(v, v') \in \mathcal{O}$, for every directed edge $(v, v') \notin E(\overline{G})$ and $(v, v') \in E(F^*)$. Clearly, $\acute{F}^*$ is obtained from $\acute{F}_{\overline{G}}$ by a series of removing and adding edges in step 2. If edge $(v, v') \notin E(\overline{G})$, then $(v, v') \notin E(\acute{F}_{\overline{G}})$. Therefore, edge (v, v') is added to graph $\acute{F}^*$ in some step 2, and $(v, v') \in \mathcal{O}$. This implies that $(\acute{F}^*, \mathcal{O}) \in A$ as desired. $\square$

Theorem 4.5. *The two maps ϕ and ψ define a bijective correspondence between sets* $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. We have shown that ϕ and ψ are well-defined. The remaining task is to prove that ϕ is the inverse map of ψ.

Given a pair $\left(\acute{F}_{K_n}, \mathcal{O}_{G,\acute{F}_{K_n},C}\right) \in \mathbf{A}$, we apply the map ϕ and obtain an output $(\acute{F}, \mathcal{W}) \in \mathbf{B}$. Suppose that during the map ϕ, we record the largest vertex u^* in every step 3 into a sequence U in order. It is easy to see that $|U| = |\mathcal{W}| = |S| - 1$, where S is the original set before the WHILE loop in map ϕ. Let $|S| = k$, and we set $U = \{u_1, \ldots, u_{k-1}\}$ and $\mathcal{W} = \{w_1, \ldots, w_{k-1}\}$. Thus, for any $1 \le j \le k-1$, (u_j, w_j) is the directed edge removed from $\acute{F}_{\overline{G}}$ in step 3 in the j-th WHILE loop.

Now, let us apply the map ψ on pair $(\acute{F}, W) \in \mathbf{B}$, and denote the output pair by $\left(\acute{F}'_{K_n}, \mathcal{O}_{G,\acute{F}'_{K_n},C_1}\right) \in \mathbf{A}$. Therefore, initially, $R = S$ is the set of roots of forest F. Our goal is to prove that

$$\acute{F}_{K_n} = \acute{F}'_{K_n} \text{ and } \mathcal{O}_{G,\acute{F}_{K_n},C_1} = \mathcal{O}_{G,\acute{F}'_{K_n},C_1}. \tag{4.1}$$

We record the vertex u we picked in every step 1 in the map ψ and get a sequence $U' = \{u'_1, \ldots, u'_{k-1}\}$ in order. Clearly, if U and U' are the same sequence, Eq.(4.1) holds since every move in step 2 in ψ will be the reverse move in step 4 in ϕ.

Before we show that $U = U'$, we first prove the following property:

(1) In the i-th WHILE loop of the map ϕ, where $1 \le i \le k-1$, consider the graph $\acute{F}_{K_n}$ after step 2. Then for any u in that current set S, it is not a leaf in $\acute{F}_{K_n}$ iff there exists some w_{i_1}, where $i \le i_1 \le k-1$, such that w_{i_1} is u or a child of u.

If u is not a leaf in $\acute{F}_{K_n}$, then there must be a vertex u' in current set S that is child of u. Consider the vertex w' which edge (u', w') is in $E(\acute{F}_{K_n})$. Consequently, $w' \in \{w_i, \ldots, w_{k-1}\}$ is vertex u or child or u. By some easy arguments, one can see that the reverse statement is true, and thus prove property (1).

We now show $U = U'$ by induction on the index i, where $1 \le i \le k-1$. When $i = 1$, clearly, from (1), we know that u'_1 is a leaf in $\acute{F}_{K_n}$. By the choice of u_1, we have $u'_1 \le u_1$. On the other hand, since u'_1 is the largest element in S that no element in $\mathcal{W}$ is u'_1 or child of u'_1, we have $u_1 \le u'_1$. As a result, $u_1 = u'_1$.

Secondly, suppose for i from 1 to $r-1$, where $r \le k-1$, we have $u_i = u'_i$. That is to say, the set S and R in the r-th WHILE loop of map ϕ and ψ are the same. When $i = r$, from (1) and the choice of u'_r, we have that both $u_r \in S$ and $u'_r \in R = S$ are the largest vertex z such that no element $w \in \{w_r, \ldots, w_{k-1}\}$ is z or child of z. Consequently, $u_r = u'_r$.

By induction, we can prove that U and U' are the same sequence. Therefore, Eq.(4.1) holds and ψ is the inverse map of ϕ. Finally, this shows us that the two maps ϕ and ψ define a coorespondence relation between sets $\mathbf{A}$ and $\mathbf{B}$. □

In particular, consider the case that $G = K_n$. Since $\overline{G}$ is empty, we have that every valid pair (u, v) in $\acute{F}_{K_n}$ is not in $\overline{G}$. Therefore, for every spanning tree $\acute{F}_{K_n}$ in $\widetilde{K_n}$, there is only one possible operational set $\mathcal{O}_{K_n,\acute{F}_{K_n},C} = Z_{K_n,\acute{F}_{K_n}}$. In addition, there is only one spanning tree $\acute{F}_{\overline{G}}$ which is the graph with every vertex connected to root 0. Consequently, for every pair $(\acute{F}_{\overline{G}}, \mathcal{W}_{\acute{F}_{\overline{G}}}) \in B$, we have that $|\mathcal{W}_{\acute{F}_{\overline{G}}}| = n-1$. That is to say, every element in $\mathbf{B}$ is associated to a sequence of length $n-1$. One

can easily see that the map ϕ now is a prufer coding for spanning trees in K_{n+1} and therefore, prufer coding is a special case for this bijection.

5. A New Proof of The Reciprocity Theorem

In this section, we show how to use this bijection to prove the reciprocity theorem.

Theorem 5.1. *Let G be a graph on the set of vertices $\{1,\ldots,n\}$. Then*

$$f_G(x;x_1,\ldots,x_n) = (-1)^{n-1}\cdot f_{\overline{G}}(-x-x_1-\cdots-x_n;x_1,\ldots,x_n). \tag{5.1}$$

Proof. First of all, we show that

$$(-1)^{n-1}\cdot f_G(x;x_1,\ldots,x_n) = f_G(-x;-x_1,\ldots,-x_n). \tag{5.2}$$

If we can show that the degree of every monomial in $f_G(x;x_1,\ldots,x_n)$ is $n-1$, then Eq.(5.2) will be true. Note that each monomial in $f_G(x;x_1,\ldots,x_n)$ corresponds to some spanning tree $\acute{F}_{\widetilde{K_n}}$ of $\widetilde{K_n}$, and we have

$$\deg\left(m\left(\acute{F}_{\widetilde{K_n}}\right)\right) = \sum_{v\in\{0,\ldots,n\}}\big(\deg(v)-1\big) = \sum_{v\in\{0,\ldots,n\}}\deg(v)-(n+1) \tag{5.3}$$

$$= 2|E|-(n+1) = n-1. \tag{5.4}$$

This implies that Eq.(5.2) is true.

Now, we show that

$$f_G(x;x_1,\ldots,x_n) = f_{\overline{G}}(x+x_1+\cdots+x_n;-x_1,\ldots,-x_n). \tag{5.5}$$

Consider some spanning tree $\acute{F}_{K_n}$ of $\widetilde{K_n}$ associated to a monomial $x^d x_1^{d_1}\cdots x_n^{d_n}$ in polynomial f_G and an operational set $\mathcal{O}_{G,\acute{F}_{K_n},C}$ for $\acute{F}_{K_n}$. Let us apply the map ϕ on $\left(\acute{F}_{K_n},\mathcal{O}_{G,\acute{F}_{K_n},C}\right)$. Denote the output pair by $\left(\acute{F}_{\overline{G}},\mathcal{W}_{\acute{F}_{\overline{G}}}\right)\in\mathbf{B}$, where sequence $\mathcal{W}_{\acute{F}_{\overline{G}}}=(w_1,\ldots,w_{k-1})$, and k is the number of connected components in $F_{\overline{G}}$. Moreover, the contribution of graph $\acute{F}_{\overline{G}}$ in the polynomial $f_{\overline{G}}$ is

$$(x+x_1+\cdots+x_n)^{k-1}(-x_1)^{\deg(v_1)-1}\cdots(-x_n)^{\deg(v_n)-1}, \tag{5.6}$$

where $\deg(v_i)$ is the degree of vertex $i\neq 0$ in $\acute{F}_{\overline{G}}$. We associate the pair $\left(\acute{F}_{\overline{G}},\mathcal{W}_{\acute{F}_{\overline{G}}}\right)$ to the monomial

$$x_{w_1}\cdots x_{w_{k-1}}(-x_1)^{\deg(v_1)-1}\cdots(-x_n)^{\deg(v_n)-1}$$

in (5.6), where $x_0=x$ and x_{w_j} is the variable corresponding to vertex w_j, for $1\leq j\leq k-1$. Clearly, $x_{w_1}\cdots x_{w_{k-1}}$ is a monomial in $(x+x_1+\cdots+x_n)^{k-1}$. By the choice of $\mathcal{W}_{\acute{F}_{\overline{G}}}$ shown in section 3, we have that the set $\mathbf{B}$ and set of all monomials in $f_{\overline{G}}(x+x_1+\cdots+x_n;-x_1,\ldots,-x_n)$ have a bijective coorespondence.

It is easy to show that the monomial for the pair $\left(\acute{F}_{K_n},\mathcal{O}_{G,\acute{F}_{K_n},C}\right)$ is the monomial associated to the pair $\left(\acute{F}_{\overline{G}},\mathcal{W}_{\acute{F}_{\overline{G}}}\right)$ with several sign changes, where the number of sign changes is $\sum_{i=1}^{n}(\deg(v_i)-1)$. That is to say, we have

$$x^d x_1^{d_1}\cdots x_n^{d_n} = (-1)^l\cdot x_{w_1}\cdots x_{w_{k-1}}x_1^{\deg(v_1)-1}\cdots x_n^{\deg(v_n)-1}, \tag{5.7}$$

where $l = \sum_{i=1}^{n}(\deg(v_i) - 1) = n - \deg(v_0)$.

Now, suppose that $\acute{F}_{K_n} \in \mathcal{T}(\widetilde{G})$. Since every valid pair in $\acute{F}_{K_n}$ is not in graph $\overline{G}$, the only operational set for $\acute{F}_{K_n}$ is $\mathbf{Z}_{G,\acute{F}_{K_n}}$. In addition, the output spanning tree $\acute{F}_{\overline{G}}$ is the extended graph of empty graph. Therefore, the only pair $\left(\acute{F}_{K_n}, \mathbf{Z}_{G,\acute{F}_{K_n}}\right) \in \mathbf{A}$ for $\acute{F}_{K_n}$ is mapped to a monomial in $(x + x_1 + \cdots + x_n)^n$. This implies that the coefficient of the monomial associated to $\acute{F}_{K_n}$ is 1 in $f_{\overline{G}}(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n)$.

Secondly, if $\acute{F}_{K_n} \notin \mathcal{T}(\widetilde{G})$, then there is an edge $(u, v) \in E(F_{K_n})$ such that $(u, v) \in E(\overline{G})$. For every operational set $\mathcal{O}_{G,\acute{F}_{K_n},C}$ for $\acute{F}_{K_n}$, we consider the two operational sets:

$$\mathcal{O}_1 = \mathcal{O}_{G,\acute{F}_{K_n},C} \cup \{(u, v)\}, \text{ and } \mathcal{O}_2 = \mathcal{O}_1 \backslash \{(u, v)\} \tag{5.8}$$

Clearly, $\mathcal{O}_1$ and $\mathcal{O}_2$ are both operational sets for $\acute{F}_{K_n}$. Denote the output pair for $(\acute{F}_{K_n}, \mathcal{O}_1)$ as $(\acute{F}_1, \mathcal{W}_1)$ and the output pair for $(\acute{F}_{K_n}, \mathcal{O}_2)$ as $(\acute{F}_2, \mathcal{W}_2)$ in the map ϕ. From Eq.(5.7), one can see that the monomials associated to the two pairs $(\acute{F}_1, \mathcal{W}_1)$ and $(\acute{F}_2, \mathcal{W}_2)$ are the same. Moreover, the degrees of root 0 in $\acute{F}_1$ and $\acute{F}_2$ are differ by 1. Consequently, by (5.7), the summation of the coefficients of the monomial associated to $(\acute{F}_1, \mathcal{W}_1)$ and $(\acute{F}_2, \mathcal{W}_2)$ is 0. Finally, because we can pair up all the operational sets for $\acute{F}_{K_n}$ by (5.8), the contribution of the monomial for $\acute{F}_{K_n}$ in $f_{\overline{G}}(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n)$ is 0.

From the above argument, we conclude that the only monomials left in $f_{\overline{G}}$ after cancellation of coefficients are the monomials in $f_G(x; x_1, \ldots, x_n)$. Moreover, each monomial in f_G has coefficient 1 in $f_{\overline{G}}(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n)$. As a result, we have that $f_G(x; x_1, \ldots, x_n) = f_{\overline{G}}(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n)$, and Eq.(5.1) holds as desired. □

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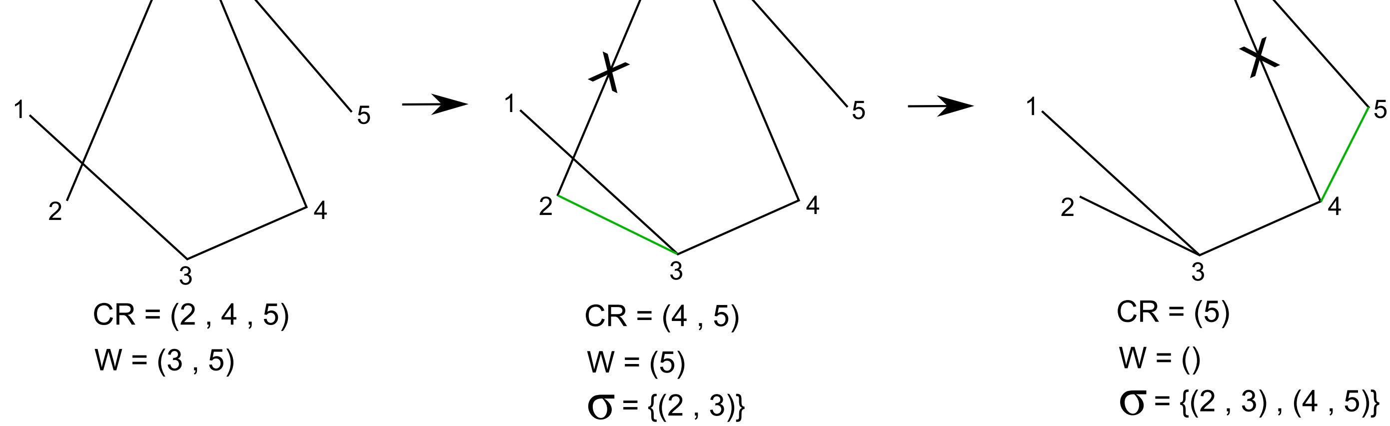
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